μsec and $r = 100$ m, the results are shown in Fig. 4. For $r/a < t < r/a + \tau$, the far-field signature, which has a finite amplitude, decreases with increasing time and approaches zero as $t \rightarrow r/a + \tau$. For $t > r/a + \tau$ the profile is the same as that in the high intensity regime and the maximum amplitude function for 5×10^5 W/cm^2 is 0.9 atm. These calculations have not taken into account the effects of frequency dependent absorption. The fast Fourier transform technique can be used to obtain such an attenuated signature.

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Moving Thermal Contact Problems

Joseph Padovan*
University of Akron, Akron, Ohio

Introduction

ALTHOUGH the problem of moving thermal interfaces has received much attention for situations[1-3] such as melting, solidification, and ablation, little work has been directed toward moving contact problems. Such problems commonly occur in rolling mills, tire-road contact, etc. The lack of work in this area is largely an outgrowth of the fact that such thermal contact problems result in multidimensional, time-dpendent, mixed boundary-value formulations which are essentially analytically intractable. Such difficulties are further aggravated for media with temperature dependent properties.[4]

With the foregoing in mind, this Note will consider the finite element (FE) solution of moving thermal interface problems where there is a steady-state formation of the contact zone. Particular emphasis will be given to contact problems associated with either closed or "infinite" structures. To simplify the overall FE formulation, rather than employing numerical integration to handle the time dependency, the Galilean transform[4] is used to convert the governing field equations to a purely spatially dependent form. Such a transformation tends to freeze the contact formation in both space and time. This operation leads to a new type of FE formulation where the overall element "conductivity" (stiffness) is no longer symmetric. Regardless of the lack of symmetry, in addition to being more tractable, the solution to this formulation is not susceptible to the instabilities and convergence difficulties of the numerical integration approach. In the sections which follow, brief discussions will be given on the governing equations and FE development as well as the results of several numerical experiments.

Governing Field Equations

For a structure composed of anisotropic temperature dependent media, the governing conduction field equation is defined by

$$\nabla \cdot ([K] \cdot \nabla T) + Q = \rho C_V \frac{\partial T}{\partial t} \tag{1}$$

where $[K]$ is the conductivity tensor, Q the heat generation, ρ the density, C_V the specific heat, T the temperature, t the time, and $\nabla \cdot (\ )$ and $\nabla (\ )$ are the divergence and gradient operators. Considering the problem of moving thermal contact, the boundary conditions associated with Eq. (1) take the form:

1) for all (x_1, x_2, x_3) on $S_{s\infty}(t)$

$$n_s \cdot [K_s] \cdot \nabla T_s = H_{s\infty}(T_s - T_\infty) \tag{2}$$

2) for all (x_1, x_2, x_3) on $S_{sg}(t)$

$$n_s \cdot [K_s] \cdot \nabla T_s = n_g \cdot [K_g] \cdot \nabla T_g \tag{3}$$

$$n_s \cdot [K_s] \cdot \nabla T_s = H_{sg}(T_s - T_g) \tag{4}$$

such that n_s, n_g are normals to bodies s and g; $H_{s\infty}$ is the convective coefficient for (x_1, x_2, x_3) on $S_{s\infty}(t)$; T_∞ is the ambient temperature in the vicinity of $S_{s\infty}(t)$ and, assuming imperfect contact, H_{sg} is the so-called contact conductance at the interface $S_{sg}(t)$.

Finite Element Development

Since Eq. (1) is time dependent and potentially nonlinear, the standard FE formulation will yield a system of first-order ordinary differential equations which must be solved numerically. Rather than handle the time dependency via direct numerical integration, since the steady-state case of moving thermal contact is assumed, the Galilean transform[4] can be used to reduce the problem to a purely spatially dependent nonlinear eigenvalue problem. As noted earlier, such an approach "freezes" the contact formation. Hence, the problem can be treated from a stationary point of view.

In particular, for the appropriate coordinate choice, the governing fields take the form namely

$$[T(x_1, x_2, x_3, t), \ldots] = [T(x_1, x_2, x_3 + \Omega t), \ldots] \tag{5}$$

where Ω is the speed of contact patch formation. For either x_3 infinite or closed† structures, the employment of the Galilean transform ($\xi = x_3 + \Omega t$) reduces Eq. (1) to the form

$$\nabla \cdot ([K] \cdot \nabla T) + Q = \rho \Omega C_V \frac{\partial}{\partial \xi}(T) \tag{6}$$

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*Associate Professor of Mechanical Engineering.

†For such a case, the governing fields are periodic in both space (x_3) and time.

In order to develop the appropriate FE field equations, the following type shape function is employed, in particular

$$T = \underset{\sim}{N}'_t (x_1, x_2, \xi) \underset{\sim}{T} \tag{7}$$

$$\underset{\sim}{q} = [B_t] \underset{\sim}{T} \tag{8}$$

where $\underset{\sim}{q}$ is the heat flux, $(\ )'$ the matrix transpose, $\underset{\sim}{T}$ the nodal temperature matrix, $\underset{\sim}{N}_t$ the thermal shape function, and

$$[B_t] = \nabla([N_t]) \tag{9}$$

Since the nonlinear time dependent version of the diffusion equation does not have a classical associated variational form, either the Galerkin's or local potential[5] approach must be used to develop the appropriate FE formulation. For the present purposes, the local potential is employed. This approach yields‡

$$\int_{V_e} \left(-\frac{1}{2} (\nabla T)' \cdot [\bar{K}] \cdot \nabla T + QT - \rho C_V \Omega \frac{\partial \bar{T}}{\partial \xi} \right) dx_1 \, dx_2 \, d\xi \rightarrow$$

$$([K] - \Omega [M])^e \underset{\sim}{T}^e + \underset{\sim}{F}^e = \underset{\sim}{b}^e \tag{10}$$

where V_e is the volume of the eth element and

$$[K]^e = \int_{V_e} [B_t]' [K] [B_t] dx_1 dx_2 d\xi \tag{11}$$

$$[M]^e = \int_{V_e} \rho C_V \underset{\sim}{N}_t \frac{\partial}{\partial \xi} (\underset{\sim}{N}'_t) dx_1 dx_2 d\xi \tag{12}$$

$$\underset{\sim}{F}^e = \int_{V_e} \underset{\sim}{N}_t Q dx_1 dx_2 d\xi \tag{13}$$

The components of $\underset{\sim}{b}$ appearing in Eq. (10) represent the net heat fluxes contributed by the eth element to its nodes. Note that, due to the incorporation of time dependence, the matrix coefficient of Eq. (10) is no longer symmetric. This is directly caused by the nonsymmetric form of $[M]^e$. After the appropriate assembly process, such nonsymmetry is also found in the global formulation. For the purely stationary case where $\Omega = 0$, Eq. (10) and its global counterpart reduce to the standard FE formulation which is symmetric.

In the case of perfect contact, T_i of the ith node is shared in common with all the surrounding elements. Hence the usual assembly process is possible, namely

$$\sum_{e_s} b^e_{si} + \sum_{e_g} b^e_{gi} = 0 \tag{14}$$

where the s and g components denote the various bodies involved in the contact zone. For the case of imperfect contact, Eq. (3) leads to an alternate type of assembly process. This follows since for a node located in the contact zone, $T_{si} \neq T_{gi}$. Thus Eq. (14) is altered to the form

$$\sum_{e_s} b^e_{si} + H^*_{sgi} (T_{si} - T_{gi}) = 0 \tag{15}$$

and

$$\sum_{e_g} b^e_{gi} + H^*_{sgi} (T_{gi} - T_{si}) = 0 \tag{16}$$

where H^*_{sgi} are appropriately proportioned values of H_{sg}. Note for large values of H^*_{sgi}, Eqs. (15) and (16) reduce essentially to Eq. (14) wherein $T_{si} \sim T_{gi}$.

‡The terms with overbars denote considerable variables.

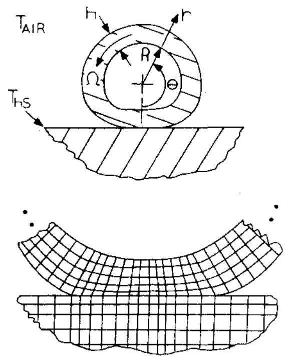

Fig. 1 Finite element simulation of roller half space contact.

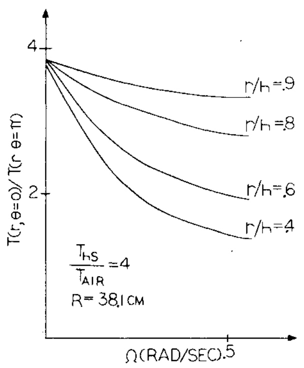

Fig. 2 Effects of rotational speed on temperature profile.

Discussion

For temperature dependent media, the global version of Eq. (10) retains its nonlinearity and for $\Omega \neq 0$, the "stiffness" its nonsymmetry. To solve the resulting set, the standard Newton's procedure can be employed wherein the resulting tangent stiffness matrix is also nonsymmetric. For the purely linear case where $[K]$ is a constant, the FE formulation can be solved via a modified Gaussian procedure wherein the storage and elimination process is amended to handle the nonsymmetry. Regardless of the additional storage requirements, such a formulation is not susceptible to the instabilities and convergence difficulties inherent to the numerical integration procedure. Furthermore, since the contact zone is stationary, its FE simulation is much simpler to develop.

To illustrate the scope of the procedure, the rubber roller-heated rigid half space thermal contact problem depicted in Fig. 1 is considered. The main mechanisms of heat release for the stated problem are essentially twofold. One consists of the rolling thermal contact. Second, heat can be released via the speed dependent convection from the exposed roller and heated half space surfaces. To handle the FE simulation of the deformity of the contacting surface, a two-dimensional quadratic isoparametrical type shape function is used for modeling purposes. Based on the FE model shown in Fig. 1, various aspects of the thermal response were considered. In performing the study, particular emphasis was given to the relationship between the speed of contact formation and the overall thermal profile in the roller. Such results are summarized in Fig. 2. Note that, although large differences in temperature occurred in the circumferential direction for $\Omega \equiv 0$, these are significantly reduced at even fairly low rates of contact formation. As would be expected, the "averaging" effect of rolling contact is most significant in the central region of the roller.

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A Finite-Difference/Galerkin Finite-Element Solution of a Turbulent Boundary Layer

Maher N. Bismarck-Nasr*
Embraer, Empresa Brasileira de Aeronáutica, S/A, São José dos Campos, São Paulo, Brazil

Nomenclature

c_f	= local skin-friction coefficient $2\tau_w/\rho u_e^2$
f	= dimensionless stream function
(M^e)	= element characteristic matrix
(M)	= system characteristic matrix
$\{m^e\}$	= element characteristic vector
$\{m\}$	= system characteristic vector
p	= static pressure
R_x	= Reynolds number xu_e/ν
u,v	= mean velocity components in x and y directions
x,y	= coordinates measured along and normal to the body
β	= pressure gradient parameter $(2\xi/u_e)$ $(du_e/d\xi)$
δ	= boundary-layer thickness
ϵ	= eddy viscosity
ϵ^+	= dimensionless eddy viscosity ϵ/ν
μ	= dynamic viscosity
ν	= kinematic viscosity
ξ,η	= transformed x,y coordinates
ρ	= fluid mass density
$-\rho\overline{uv}$	= Reynolds shear-stress term
τ	= shear stress
ψ	= streamfunction
ζ	$= (2\xi)^{1/2}/\mu$

Subscripts

e	= outer edge of boundary layer
i	= inner region
o	= outer region
w	= wall

Superscript

$'$	= derivatives with respect to η

I. Introduction

In recent years, with the advent of high-speed computers, much progress in the numerical solutions of the turbulent boundary-layer equations, in their differential form, has been made. Finite-difference schemes have been used extensively,[1] and sufficiently accurate results were obtained for two-dimensional incompressible turbulent flows,[2-4] compressible turbulent flows,[5-8] and turbulent flows with heat and mass transfer.[9,10] Boussinesq's eddy-viscosity concept and Prandtl's mixing length theory have been used widely in these methods in order to relate the Reynolds shear-stress terms to the mean velocities.

In Ref. 11, a finite-difference/Galerkin finite-element method was presented for the solution of compressible laminar flows. In the present Note, the application of the method is extended to the solution of incompressible turbulent flows. The governing equations are presented and transformed first; the application of the finite-difference method in the streamwise direction and the Galerkin finite-element method through the boundary layer then are introduced; and the method of solution is discussed. Numerical results are given for turbulent flows over a flat plate and are compared with the finite-difference solutions of other investigators.

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*Engineer.

II. Theoretical Formulation

A. Governing Equations

Neglecting the normal stress terms, the boundary-layer equations for steady, incompressible, turbulent two-dimensional flows can be written as follows[1-2]:

Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Momentum

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}+\rho\overline{uv}\right) \tag{2}$$

The boundary conditions are as follows.[12] For no slip on the wall,

$$u(x,0)=0 \tag{3a}$$

For no mass transfer on the wall or for flows with mass transfer on the wall, respectively,

$$v(x,0)=0; \quad v(x,0)=v_w(x) \tag{3b}$$

For smooth merging of the velocity into the freestream value,

$$\lim_{y\to\infty} u(x,y)=u_e(x) \tag{3c}$$

In the following, Boussinesq's eddy-viscosity concept is used in order to relate the Reynolds shear stress to the mean velocity, i.e.,

$$-\rho\overline{uv}=\rho\epsilon\frac{\partial u}{\partial y} \tag{4}$$

and, according to this concept, the turbulent boundary layer will be regarded as composed of two regions with separate expressions for ϵ in each region. In the inner region, an expression for ϵ based on Prandtl's mixing length theory, and incorporating the modifications introduced by Van Driest[13] to account for the viscous sublayer close to the wall and by Cebeci[14] to account for flows with pressure gradients, is used and reads

$$\epsilon_i=(0.4y)^2\left[1-\exp\left\{-y\left(\frac{\tau_w}{\rho}+\frac{dp}{dx}\frac{y}{\rho}\right)\Big/26\nu\right\}\right]^2\left|\frac{\partial u}{\partial y}\right| \tag{5}$$